

**Weierstraß-Institut**  
**für Angewandte Analysis und Stochastik**  
**Leibniz-Institut im Forschungsverbund Berlin e. V.**

Preprint

ISSN 0946 – 8633

**Parabolic Anderson model**  
**with a finite number of moving catalysts**

Fabienne Castell<sup>1</sup>, Onur Gün<sup>2</sup>, Gregory Maillard<sup>1,3</sup>

submitted: 15 Dec 2011

<sup>1</sup> CMI-LATP  
Université de Provence  
39 rue F. Joliot-Curie  
13453 Marseille Cedex 13  
France  
E-Mail: castell@cmi.univ-mrs.fr

<sup>2</sup> Weierstrass Institute  
Mohrenstr. 39  
10117 Berlin  
Germany  
E-Mail: onur.guen@wias-berlin.de

<sup>3</sup> EURANDOM  
P.O. Box 513  
5600 MB Eindhoven  
The Netherlands  
E-Mail: maillard@cmi.univ-mrs.fr

No. 1669  
Berlin 2011



---

2010 *Mathematics Subject Classification.* 60H25, 82C44, 60F10, 35B40.

*Key words and phrases.* Parabolic Anderson problem, catalytic random medium, intermittency, moment Lyapunov exponents.

Edited by  
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)  
Leibniz-Institut im Forschungsverbund Berlin e. V.  
Mohrenstraße 39  
10117 Berlin  
Germany

Fax: +49 30 2044975  
E-Mail: [preprint@wias-berlin.de](mailto:preprint@wias-berlin.de)  
World Wide Web: <http://www.wias-berlin.de/>

**Abstract.** We consider the parabolic Anderson model (PAM) which is given by the equation  $\partial u / \partial t = \kappa \Delta u + \xi u$  with  $u: \mathbb{Z}^d \times [0, \infty) \rightarrow \mathbb{R}$ , where  $\kappa \in [0, \infty)$  is the diffusion constant,  $\Delta$  is the discrete Laplacian, and  $\xi: \mathbb{Z}^d \times [0, \infty) \rightarrow \mathbb{R}$  is a space-time random environment. The solution of this equation describes the evolution of the density  $u$  of a “reactant”  $u$  under the influence of a “catalyst”  $\xi$ .

In the present paper we focus on the case where  $\xi$  is a system of  $n$  independent simple random walks each with step rate  $2d\rho$  and starting from the origin. We study the *annealed* Lyapunov exponents, i.e., the exponential growth rates of the successive moments of  $u$  w.r.t.  $\xi$  and show that these exponents, as a function of the diffusion constant  $\kappa$  and the rate constant  $\rho$ , behave differently depending on the dimension  $d$ . In particular, we give a description of the intermittent behavior of the system in terms of the annealed Lyapunov exponents, depicting how the total mass of  $u$  concentrates as  $t \rightarrow \infty$ . Our results are both a generalization and an extension of the work of Gärtner and Heydenreich [3], where only the case  $n = 1$  was investigated.

## 1 Introduction

### 1.1 Model

The parabolic Anderson model (PAM) is the partial differential equation

$$\begin{cases} \frac{\partial}{\partial t} u(x, t) = \kappa \Delta u(x, t) + \xi(x, t) u(x, t), & x \in \mathbb{Z}^d, t \geq 0. \\ u(x, 0) = 1, \end{cases} \quad (1)$$

Here, the  $u$ -field is  $\mathbb{R}$ -valued,  $\kappa \in [0, \infty)$  is the diffusion constant,  $\Delta$  is the discrete Laplacian acting on  $u$  as

$$\Delta u(x, t) = \sum_{\substack{y \in \mathbb{Z}^d \\ y \sim x}} [u(y, t) - u(x, t)]$$

( $y \sim x$  meaning that  $y$  is nearest neighbor of  $x$ ), and

$$\xi = (\xi_t)_{t \geq 0} \quad \text{with} \quad \xi_t = \{\xi(x, t) : x \in \mathbb{Z}^d\}$$

is an  $\mathbb{R}$ -valued random field that evolves with time and that drives the equation.

One interpretation of (1) comes from population dynamics by considering a system of two types of particles  $A$  and  $B$ .  $A$  particles represent “catalysts”,  $B$  particles represent “reactants” and the dynamics is subject to the following rules:

- $A$ -particles evolve independently of  $B$ -particles according to a prescribed dynamics with  $\xi(x, t)$  denoting the number of  $A$ -particles at site  $x$  at time  $t$ ;

- $B$ -particles perform independent simple random walks at rate  $2d\kappa$  and split into two at a rate that is equal to the number of  $A$ -particles present at the same location;
- the initial configuration of  $B$ -particles is that there is exactly one particle at each lattice site.

Then, under the above rules,  $u(x, t)$  represents the average number of  $B$ -particles at site  $x$  at time  $t$  conditioned on the evolution of the  $A$ -particles.

It is possible to add that  $B$ -particles die at rate  $\delta \in [0, \infty)$ . This leads to the trivial transformation  $u(x, t) \rightarrow u(x, t)e^{-\delta t}$ . We will hereafter assume that  $\delta = 0$ . It is also possible to add a coupling constant  $\gamma \in (0, \infty)$  in front of the  $\xi$ -term in (1), but this can be reduced to  $\gamma = 1$  by a scaling argument.

In what follows, we focus on the case where

$$\xi(x, t) = \sum_{k=1}^n \delta_x(Y_k^p(t)) \quad (2)$$

with  $\{Y_k^p : 1 \leq k \leq n\}$  is a family of  $n$  independent simple random walks, where for each  $k \in \{1, \dots, n\}$ ,  $Y_k^p = (Y_k^p(t))_{t \geq 0}$  is a simple random walk with step rate  $2d\rho$  starting from the origin. We write  $\mathbb{P}_0^{\otimes n}$  and  $\mathbb{E}_0^{\otimes n}$  to denote respectively the law and the expectation of the family of  $n$  independent simple random walks  $\{Y_k^p : 1 \leq k \leq n\}$  where initially all of the walkers are located at 0.

Under this choice of catalysts, we will prove existence and derive both qualitative and quantitative properties of the annealed Lyapunov exponents (defined in Section 1.2). After that, we will discuss the intermittent behavior of the solution  $u$  of the PAM in terms of the Lyapunov exponents.

## 1.2 Lyapunov exponents and intermittency

Our focus will be on the *annealed* Lyapunov exponents that describes the exponential growth rate of the successive moments of the solution of (1).

By the Feynman-Kac formula, the solution of (1) reads

$$u(x, t) = \mathbb{E}_x \left( \exp \left[ \int_0^t \xi(X^\kappa(s), t-s) ds \right] \right), \quad (3)$$

where  $X^\kappa = (X^\kappa(t))_{t \geq 0}$  is the simple random walk on  $\mathbb{Z}^d$  with step rate  $2d\kappa$  and  $\mathbb{E}_x$  denotes expectation with respect to  $X^\kappa$  given  $X^\kappa(0) = x$ . The connection between the parabolic Anderson equation (1) with random time-independent potential  $\xi$  and the Feynman-Kac functional (3) is well understood (see e.g. Gärtner and Molchanov [10]) and can be easily extended to the time-dependent potential setting. Taking into account our choice of catalytic medium in (2) we define  $\Lambda_p(t)$  as

$$\begin{aligned}
\Lambda_p(t) &= \frac{1}{t} \log \mathbb{E}_0^{\otimes n} ([u(x, t)]^p)^{1/p} \\
&= \frac{1}{pt} \log (\mathbb{E}_0^{\otimes n} \otimes \mathbb{E}_x^{\otimes p}) \left( \exp \left[ \sum_{j=1}^p \sum_{k=1}^n \int_0^t \delta_0(X_j^\kappa(s) - Y_k^p(t-s)) ds \right] \right), \quad (4)
\end{aligned}$$

where  $\{X_j^\kappa, j = 1, \dots, p\}$  is a family of  $p$  independent copies of  $X^\kappa$  and  $\mathbb{E}_x^{\otimes p}$  stands for the expectation of this family with  $X_j^\kappa(0) = x$  for all  $j$ .

If the last quantity admits a limit as  $t \rightarrow \infty$  we define

$$\lambda_p := \lim_{t \rightarrow \infty} \Lambda_p(t) \quad (5)$$

to be the  $p$ -th (annealed) Lyapunov exponent of the solution  $u$  of the parabolic Anderson problem (1).

We will see in Theorem 1.1 that the limit in (5) exists and is independent of  $x$ . Hence, we suppress  $x$  in the notation. However,  $\lambda_p$  is clearly a function of  $n, d, \kappa$  and  $\rho$ . In what follows, our main focus will be to analyze the dependence of  $\lambda_p$  on the parameters  $n, p, \kappa$  and  $\rho$ , therefore we will often write  $\lambda_p^{(n)}(\kappa, \rho)$ .

In particular, our main subject of interest will be to draw the qualitative picture of *intermittency* for these systems. First, note that by the moment inequality we have

$$\lambda_p^{(n)} \geq \lambda_{p-1}^{(n)}, \quad (6)$$

for all  $p \in \mathbb{N} \setminus \{1\}$ . The system (or the solution of the system) (1) is said to be *p-intermittent* if the above inequality is strict, namely,

$$\lambda_p^{(n)} > \lambda_{p-1}^{(n)}. \quad (7)$$

The system is *fully intermittent* if (7) holds for all  $p \in \mathbb{N} \setminus \{1\}$ . We will sometimes say that the system is *partially intermittent* if it is  $p$ -intermittent for some  $p \in \mathbb{N} \setminus \{1\}$ .

Also note that, using Hölder's inequality,  $p$ -intermittency implies  $q$ -intermittency for all  $q \geq p$  (see e.g. [3], Lemma 3.1). Thus, for any fixed  $n \in \mathbb{N}$ ,  $p$ -intermittency in fact implies that

$$\lambda_q^{(n)} > \lambda_{q-1}^{(n)} \quad \forall q \geq p,$$

and 2-intermittency means full intermittency.

Geometrically, intermittency corresponds to the solution being *asymptotically concentrated* on a thin set, which is expected to consist of “islands” located far from each other (see [9], Section 1 and references therein for more details). Here, due to the lack of ergodicity, such a geometric picture of intermittency is not available. Nevertheless, (7) can still be interpreted as the  $p$ th moment of  $u$  being generated by some exponentially rare event (see [3], Section 1.2 for a more detailed analysis).

### 1.3 Literature

The behavior of the *annealed* Lyapunov exponents and particularly the problem of intermittency for the PAM in a *space-time* random environment was subject to various studies. Carmona and Molchanov [2] obtained an essentially complete qualitative description of the annealed Lyapunov exponents and intermittency when  $\xi$  is white noise, i.e.,

$$\xi(x, t) = \frac{\partial}{\partial t} W(x, t),$$

where  $W = (W_t)_{t \geq 0}$  with  $W_t = \{W(x, t) : x \in \mathbb{Z}^d\}$  is a field of independent Brownian motions. Further refinements on the behavior of the Lyapunov exponents were obtained in Greven and den Hollander [11]. In particular, it was shown that  $\lambda_1 = 1/2$  for all  $d \geq 1$  and,  $\lambda_p > 1/2$  for  $p \in \mathbb{N} \setminus \{1\}$  in  $d = 1, 2$ . It is also proved that for  $d \geq 3$  there exist  $0 < \kappa_2 \leq \kappa_3 \leq \dots$  satisfying

$$\lambda_p(\kappa) - \frac{1}{2} \begin{cases} > 0, & \text{for } \kappa \in [0, \kappa_p), \\ = 0, & \text{for } \kappa \in [\kappa_p, \infty), \end{cases} \quad p \in \mathbb{N} \setminus \{1\}.$$

Upper and lower bounds on  $\kappa_p$  were derived, and the asymptotics of  $\kappa_p$  as  $p \rightarrow \infty$  was computed. In addition, it was proved that the  $\kappa_p$ 's are distinct for  $d$  large enough.

More recently various models where  $\xi$  is *non-Gaussian* were investigated. Kesten and Sidoravicius [13] and Gärtner and den Hollander [4] considered the case where  $\xi$  is given by a Poisson field of independent simple random walks. In [13], the survival versus extinction of the system is studied. In [4], the moment asymptotics were studied and a partial picture of intermittency, depending on the parameters  $d$  and  $\kappa$ , was obtained. The case where  $\xi$  is a single random walk –corresponding to  $n = 1$  case in our setting– was studied by Gärtner and Heydenreich [3]. Analogous results to those contained in Theorems 1.1, 1.2 and Corollary 1.1(i) were obtained.

The investigation of annealed Lyapunov behavior and intermittency was extended to non-Gaussian and *space correlated* potentials in Gärtner, den Hollander and Maillard, in [5] and [7], for the case where  $\xi$  is an exclusion process with symmetric random walk transition kernel, starting from a Bernoulli product measure. Later in Gärtner, den Hollander and Maillard [8], the case where  $\xi$  is a voter model starting either from Bernoulli product measure or from equilibrium was studied (see Gärtner, den Hollander and Maillard [6], for an overview).

### 1.4 Main results

Our first theorem states that the Lyapunov exponents exist and behave nicely as a function of  $\kappa$  and  $\rho$ . It will be proved in Section 2.

**Theorem 1.1 (Existence and first properties).** *Let  $d \geq 1$  and  $n, p \in \mathbb{N}$ .*

*(i) For all  $\kappa, \rho \in [0, \infty)$ , the limit in (5) exists, is finite, and is independent of  $x$  if*

$(\kappa, \rho) \neq (0, 0)$ .

(ii) On  $[0, \infty)^2$ ,  $(\kappa, \rho) \mapsto \lambda_p^{(n)}(\kappa, \rho)$  is continuous, convex and non-increasing in both  $\kappa$  and  $\rho$ .

Let  $G_d(x)$  be the Green function at lattice site  $x$  of simple random walk stepping at rate  $2d$  and

$$\mu(\kappa) = \sup \text{Sp}(\kappa \Delta + \delta_0) \quad (8)$$

be the supremum of the spectrum of the operator  $\kappa \Delta + \delta_0$  in  $l^2(\mathbb{Z}^d)$ . It is well-known that (see e.g. [4], Lemma 1.3)  $\text{Sp}(\kappa \Delta + \delta_0) = [-4d\kappa, 0] \cup \{\mu(\kappa)\}$  with

$$\mu(\kappa) \begin{cases} = 0, & \text{if } \kappa \geq G_d(0), \\ > 0, & \text{if } \kappa < G_d(0). \end{cases} \quad (9)$$

Furthermore,  $\kappa \mapsto \mu(\kappa)$  is continuous, non-increasing and convex on  $[0, \infty)$ , and strictly decreasing on  $[0, G_d(0)]$ .

The next theorem gives the limiting behavior of  $\lambda_p^{(n)}$  as  $\kappa \downarrow 0$  and  $\kappa \rightarrow \infty$ , and describes a region of  $\kappa$  where  $\lambda_p^{(n)} = 0$ . Note that by symmetry,  $\lambda_p^{(n)}(\kappa, \rho) = \frac{n}{p} \lambda_n^{(p)}(\rho, \kappa)$ , for all  $n, p \in \mathbb{N}$  and  $\kappa, \rho \in [0, \infty)$ . Therefore, the  $\kappa$ -dependence described below can be transcribed in terms of  $\rho$ -dependence.

**Theorem 1.2 ( $\kappa$ -  $\rho$ -dependence).** *Let  $n, p \in \mathbb{N}$  and  $\rho \in [0, \infty)$ .*

- (i) *For all  $d \geq 1$ ,  $\lim_{\kappa \downarrow 0} \lambda_p^{(n)}(\kappa, \rho) = \lambda_p^{(n)}(0, \rho) = n\mu(\rho/p)$ .*
- (ii) *If  $1 \leq d \leq 2$ , then  $\lambda_p^{(n)}(\kappa, \rho) > 0$  for all  $\kappa \in [0, \infty)$ . Moreover,  $\kappa \mapsto \lambda_p^{(n)}(\kappa, \rho)$  is strictly decreasing with  $\lim_{\kappa \rightarrow \infty} \lambda_p^{(n)}(\kappa, \rho) = 0$  (see Fig. 1).*
- (iii) *If  $d \geq 3$ , then  $\lambda_p^{(n)}(\kappa, \rho) = 0$  for all  $\kappa \in [nG_d(0), \infty)$  (see Fig. 2).*

Our next result describes the limiting behavior of  $\lambda_p^{(n)}$  as  $p \rightarrow \infty$  and  $n \rightarrow \infty$ .

**Theorem 1.3 ( $n$ -  $p$ -dependence).** *Let  $d \geq 1$  and  $\kappa, \rho \in [0, \infty)$ .*

- (i) *For all  $n \in \mathbb{N}$ ,  $\lim_{p \rightarrow \infty} \lambda_p^{(n)}(\kappa, \rho) = n\mu(\kappa/n)$  (see Fig. 1–2);*
- (ii) *For all  $p > \rho/G_d(0)$ ,  $\lim_{n \rightarrow \infty} \lambda_p^{(n)}(\kappa, \rho) = +\infty$ ;*
- (iii) *For all  $p \leq \rho/G_d(0)$  and  $n \in \mathbb{N}$ ,  $\lambda_p^{(n)}(\kappa, \rho) = 0$ .*

By part (ii) of Theorem 1.1,  $\lambda_p^{(n)}(\kappa, \rho)$  is non-increasing in  $\kappa$ . Hence, we can define  $\{\kappa_p^{(n)}(\rho) : p \in \mathbb{N}\}$  as the non-decreasing sequence of critical  $\kappa$ 's for which

$$\lambda_p^{(n)}(\kappa, \rho) \begin{cases} > 0, & \text{for } \kappa \in [0, \kappa_p^{(n)}(\rho)), \\ = 0, & \text{for } \kappa \in [\kappa_p^{(n)}(\rho), \infty), \end{cases} \quad p \in \mathbb{N}. \quad (10)$$

As a consequence of Theorems 1.1 and 1.2 we have,

$$\begin{cases} \kappa_p^{(n)}(\rho) = \infty, & \text{if } 1 \leq d \leq 2, \\ 0 < \kappa_p^{(n)}(\rho) < \infty, & \text{if } d \geq 3 \text{ and } p > \rho/G_d(0), \\ \kappa_p^{(n)}(\rho) = 0, & \text{if } d \geq 3 \text{ and } p \leq \rho/G_d(0). \end{cases} \quad (11)$$

Our fourth theorem, which gives bounds on  $\kappa_p^{(n)}(\rho)$  for  $d \geq 3$ , will be proved in Section 4. For this theorem we need to define the inverse of the function  $\mu(\kappa)$ . Note that by (8) and (9) we have  $\mu(0) = 1$  and  $\mu(G_d(0)) = 0$ . It is easy to see that  $\mu(\kappa)$  restricted to the domain  $[0, G_d(0)]$  is invertible with an inverse function  $\mu^{-1}: [0, 1] \rightarrow [0, G_d(0)]$ . We extend  $\mu^{-1}$  to  $[0, \infty)$  by declaring  $\mu^{-1}(t) = 0$  for  $t > 1$ . Denote

$$\alpha_d = \frac{G_d(0)}{2d\|G_d\|_2^2} \in [0, \infty), \quad (12)$$

where  $\|G_d\|_2$  is the  $l_2$  norm of  $G_d$ . Since  $\|G_d\|_2 < \infty$  if and only if  $d \geq 5$ ,  $\alpha_d = 0$  for  $d \in \{3, 4\}$ .

**Theorem 1.4 (Critical  $\kappa$ 's).** *Let  $n, p \in \mathbb{N}$ .*

(i) *If  $d \geq 3$ , then  $\rho \in [0, \infty) \mapsto \kappa_p^{(n)}(\rho)$  is a continuous, non-increasing and convex function such that*

$$\max\left(\frac{n}{4d}\mu(\rho/p), n\mu^{-1}(4d\rho/p)\right) \leq \kappa_p^{(n)}(\rho) \leq nG_d(0)\left(1 - \frac{\rho}{pG_d(0)}\right)_+. \quad (13)$$

(ii) *If  $d \geq 5$ , then*

$$\kappa_p^{(n)}(\rho) \geq \left(nG_d(0) - \rho \frac{n}{p\alpha_d}\right)_+. \quad (14)$$

(iii) *If  $d \geq 5$  and  $p \in \mathbb{N} \setminus \{1\}$  are such that  $\alpha_d > \frac{p-1}{p}$ , then*

$$\kappa_{p-1}^{(n)}(\rho) < \kappa_p^{(n)}(\rho) \quad \forall \rho \in (0, pG_d(0)). \quad (15)$$

Note that the condition  $\alpha_d > \frac{p-1}{p}$  is always true if  $d$  is large enough by the following lemma, whose proof is given in the appendix.

**Lemma 1.1.** *If  $d \geq 3$ , then  $\alpha_d \leq 1$  and  $\lim_{d \rightarrow \infty} \alpha_d = 1$ .*

As a consequence of the previous statements, our next result gives some general intermittency properties for all dimensions, and describes several regimes in the intermittent behavior of the system.

**Corollary 1.1 (Intermittency).** *Let  $n \in \mathbb{N}$ .*

(i) *If  $d \geq 1$ , then (see Fig. 2)*

- *for  $\kappa \in [0, nG_d(0))$  there exists  $p \geq 2$  such that the system is  $p$ -intermittent;*
- *for  $\kappa \in [nG_d(0), \infty)$  the system is not intermittent.*

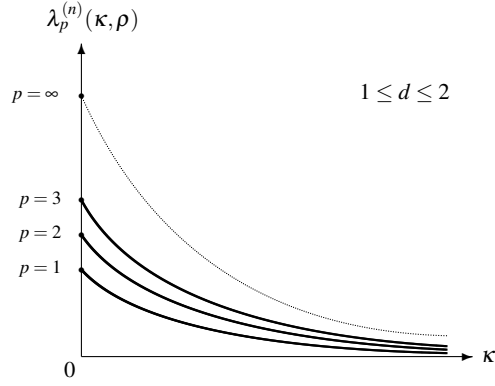
(ii) *Fix  $p \in \mathbb{N} \setminus \{1\}$ . If  $d$  is large enough (such that  $\alpha_d > (p-1)/p$ ), then (see Fig. 3):*

- *for  $\rho \in (0, 2G_d(0))$  and  $\kappa \in [\kappa_1^{(n)}(\rho), \kappa_2^{(n)}(\rho))$ , the system is 2-intermittent;*
- *for  $\rho \in [0, 3G_d(0))$  and  $\kappa \in (\kappa_2^{(n)}(\rho), \kappa_3^{(n)}(\rho))$ , the system is 3-intermittent;*
- *...*

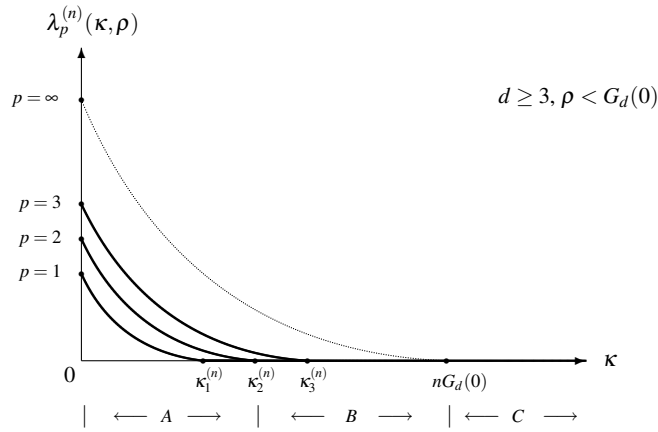


- for  $\rho \in [0, pG_d(0))$  and  $\kappa \in (\kappa_{p-1}^{(n)}(\rho), \kappa_p^{(n)}(\rho))$  the system is  $p$ -intermittent.

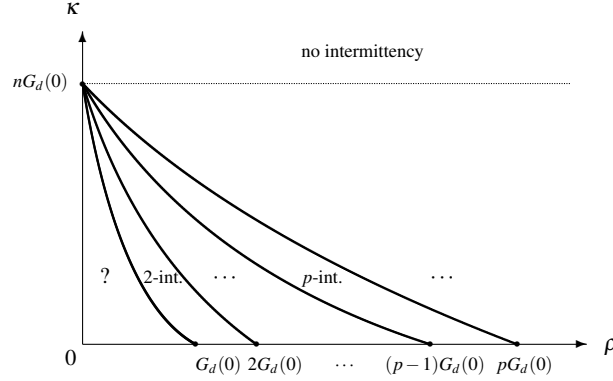
Note that since  $G_d(0) = \infty$  for  $d = 1, 2$  Corollary 1.1(i) implies that for dimensions 1 and 2 the system is always  $p$ -intermittent for some  $p$ . Some other partial results about intermittency are given in section 1.5 (see also figures).



**Fig. 1** For  $1 \leq d \leq 2$ , the system is partially intermittent. Full intermittency is conjectured, and proved for  $n = 1, 2$ .



**Fig. 2** For  $d \geq 3$  and  $\rho < G_d(0)$ , the system is partially intermittent on  $A \cup B$  and not intermittent on  $C$ . Full intermittency on  $A$  is conjectured, and proved for  $n = 1, 2$ .



**Fig. 3** Phase diagram of intermittency when  $d$  is large enough. The bold curves represent  $\rho \in [0, \infty) \mapsto \kappa_q^{(n)}(\rho)$ ,  $q = 1, \dots, p$ . In the “?” region, full intermittency is proved in a small neighborhood of 0.

### 1.5 Discussion

Our results can be extended to various different random medium. For example, consider the system of catalysts given by a collection of independent random walks where there is one walker starting from each site of a large box. More precisely, let  $D_R$  denote the box in  $\mathbb{Z}^d$  with side length  $R$ . Consider the random medium

$$\xi(x, t) = \sum_{k \in D_R} \delta_x(Y_k^\rho(t))$$

with  $\{Y_k^\rho : k \in D_R\}$  a family of  $R^d$  simple random walks, where for each  $k \in D_R$ ,  $Y_k^\rho$  is a simple random walk with step rate  $2d\rho$  starting from  $Y_k^\rho(0) = k$ . For a fixed size box, there is a positive probability that all the random walks meet at the origin in finite time. Then, it is easy to see that the Lyapunov exponents are the same as in the case of  $n$  independent random walks starting from the origin where  $n = R^d$ . An interesting set up would be case where the length of the initial box grows with time. A natural question arises as whether the large time limit would be related to the case of Poisson field of simple random walks, considered in [4], or it would have different behavior depending on how fast the size of the box grows with time.

Let us now discuss some facts about the intermittent picture. First of all, as one can easily guess from (4),  $\lambda_p^{(n)}(\kappa, \rho)$  is the top of the spectrum of the operator  $\mathcal{L}_\rho$  where for  $f(x_1, \dots, x_p, y_1, \dots, y_n)$  in  $l_2(\mathbb{Z}^{d(p+n)})$   $\mathcal{L}_\rho$  is defined by:

$$\mathcal{L}_\rho(f) = \kappa \sum_{k=1}^p \Delta_{x_k} f + \rho \sum_{j=1}^n \Delta_{y_j} f + I_p f. \quad (16)$$

Here

$$I_p f(x_1, \dots, x_p, y_1, \dots, y_n) = \sum_{k=1}^n \sum_{j=1}^p \delta_0(x_j - y_k) f(x_1, \dots, x_p, y_1, \dots, y_n).$$

This is the meaning of equation (20) of Section 2 from which most of our results are derived. The following proposition links full intermittency and existence of an eigenfunction corresponding to  $\lambda_1^{(n)}(\kappa, \rho)$ .

**Proposition 1.1.** *If there exists  $f \in l_2(\mathbb{Z}^{d(1+n)})$  with  $\|f\|_2 = 1$ , such that  $\mathcal{L}_1(f) = \lambda_1^{(n)}(\kappa, \rho)f$  then  $\lambda_2^{(n)}(\kappa, \rho) > \lambda_1^{(n)}(\kappa, \rho)$ , and the system is fully intermittent.*

Proposition 1.1 is proved in the appendix. The existence of an eigenfunction corresponding to  $\lambda_1^{(n)}(\kappa, \rho)$  (and therefore full intermittency) was proved in the following cases:

- $n = 1, 2$  and  $\kappa + \rho < nG_d(0)$ . This is done in [3] for  $n = 1$ , and in [14] for  $n = 2$ .
- $n \geq 3$  and  $4d(\rho n + \kappa) < 1$  in [14].

To prove these results, the papers [3] and [14] express  $\lambda_1^{(n)}(\kappa, \rho)$  as the top of the spectrum of the operator  $\mathcal{H} = \mathcal{B} + \sum_{j=1}^n \delta_0(z_j)$ , where  $\mathcal{B}$  is the generator of the Markov process  $Z(t) = (X_1^\kappa(t) - Y_1^\rho(t), \dots, X_1^\kappa(t) - Y_n^\rho(t))$  (see (4)). For  $n = 1$ ,  $\mathcal{H}$  is just  $(\kappa + \rho)\Delta + \delta_0$ , which is a compact perturbation of  $(\kappa + \rho)\Delta$ . This fact easily implies the existence of an eigenfunction corresponding to  $\lambda_1^{(1)}(\kappa, \rho)$  easily. However, this is no more the case as soon as  $n \geq 2$ . In [14], the authors consider  $\mathcal{B}$  as a perturbation of  $\sum_{j=1}^n \delta_0(z_j)$ , leading to the results for  $n \geq 2$ . Expressing  $\lambda_p^{(n)}(\kappa, \rho)$  in terms of the process  $(Z(t), t \geq 0)$  does not seem very fruitful in cases others than the ones treated in [3] and [14]. Therefore, it appeared to us more tractable and more natural to express  $\lambda_p^{(n)}(\kappa, \rho)$  in terms of the process  $(X_1^\kappa(t), \dots, X_p^\kappa(t), Y_1^\rho(t), \dots, Y_n^\rho(t))$ .

We complete the intermittent picture by the following conjecture

*Conjecture 1.1 (Intermittency).* Fix  $n \in \mathbb{N}$ . Then (see Fig. 1–2),

- (i) for  $1 \leq d \leq 2$ , the system is full intermittent (proved for  $n = 1, 2$ );
- (ii) for  $d \geq 3$ , the intermittency vanishes as  $\kappa$  increases. More precisely, for  $d \geq 3$ , there are three different regimes:

- A: for  $\kappa \in [0, \kappa_2^{(n)})$ , the system is full intermittent (proved in a small neighborhood of 0);
- B: for  $\kappa \in [\kappa_2^{(n)}, nG_d(0))$ , there exists  $p = p(\kappa) \geq 3$  such that the system is  $q$ -intermittent for all  $q \geq p$ ;
- C: for  $\kappa \in [nG_d(0), \infty)$ , the system is not  $p$ -intermittent for any  $p \geq 2$ .

To complete Theorem 1.4, we close with a conjecture about critical  $\kappa$ 's, whose analogue for white noise potential was conjectured in Carmona and Molchanov [2] and partially proved in Greven and den Hollander [11]:

*Conjecture 1.2 (Critical  $\kappa$ 's).* For all fixed  $n \geq 1$  and  $d$  large enough the  $\kappa_p^{(n)}$ 's are distinct (see Fig. 2).

## 2 Proof of Theorem 1.1

**Step 1:** We first prove that if the limit in (5) exists for  $x = 0$ , then it exists for all  $x \in \mathbb{Z}^d$  and does not depend on  $x$  as soon as  $(\kappa, \rho) \neq (0, 0)$ . To this end, let us introduce some notations. For any  $t > 0$ , we denote

$$Y_t = (Y_1^\rho(t), \dots, Y_n^\rho(t)) \in \mathbb{Z}^{dn}, X_t = (X_1^\kappa(t), \dots, X_p^\kappa(t)) \in \mathbb{Z}^{dp}.$$

For  $(x, y) \in \mathbb{Z}^{dp} \times \mathbb{Z}^{dn}$ ,  $\mathbb{E}_{x,y}^{X,Y}$  denote the expectation under the law of  $(X_t, Y_t)_{t \geq 0}$  starting from  $(x, y)$ . The same notation is used for  $x \in \mathbb{Z}^d$  and  $y \in \mathbb{Z}^d$ . In that case, it means that  $X_0 = (x, \dots, x)$ ,  $Y_0 = (y, \dots, y)$  and  $\mathbb{E}_{x,y}^{X,Y} = \mathbb{E}_y^{\otimes n} \otimes \mathbb{E}_x^{\otimes p}$ . Finally, for  $x = (x_1, \dots, x_p) \in \mathbb{Z}^{dp}$  and  $y = (y_1, \dots, y_n) \in \mathbb{Z}^{dn}$ , set

$$I_p(x, y) = \sum_{j=1}^p \sum_{k=1}^n \delta_0(x_j - y_k). \quad (17)$$

Then, by time reversal for  $Y$  in (4), for all  $x \in \mathbb{Z}^d$  and  $t > 0$ ,

$$\mathbb{E}_0^{\otimes n} [u(x, t)^p] = \sum_{z \in \mathbb{Z}^{dn}} \mathbb{E}_{x,z}^{X,Y} \left[ \exp \left( \int_0^t I_p(X_s, Y_s) ds \right) \delta_0(Y_t) \right]. \quad (18)$$

Using the Markov property at time 1 and the fact that  $1 \leq \exp \left( \int_0^1 I_p(X_s, Y_s) ds \right)$ , we get for  $x_1$  and  $x_2$  any fixed points in  $\mathbb{Z}^d$ ,

$$\begin{aligned} \mathbb{E}_0^{\otimes n} [u(x_1, t)^p] &\geq \sum_{z \in \mathbb{Z}^{dn}} \mathbb{E}_{x_1, z}^{X,Y} \left[ \delta_{(x_2, \dots, x_2)}(X_1) \delta_z(Y_1) \exp \left( \int_1^t I_p(X_s, Y_s) ds \right) \delta_0(Y_t) \right] \\ &= (p_1^\kappa(x_1, x_2))^p (p_1^\rho(0, 0))^n \mathbb{E}_0^{\otimes n} ([u(x_2, t-1)]^p), \end{aligned}$$

where  $p_t^\nu$  is the transition kernel of a simple random walk on  $\mathbb{Z}^d$  with step rate  $2d\nu$ . This proves the independence of  $\lambda_p$  w.r.t.  $x$  as soon as  $\kappa > 0$ , since in this case for all  $x_1, x_2 \in \mathbb{Z}^d$ ,  $p_1^\kappa(x_1, x_2) > 0$ .

For  $\kappa = 0$ , since the  $X$ -particles do not move, we have

$$\mathbb{E}_0^{\otimes n} [u(x_1, t)^p] = \mathbb{E}_0 \left[ \exp \left( p \int_0^t \delta_{x_1}(Y_1^\rho(s)) ds \right) \right]^n. \quad (19)$$

The same reasoning leads now to

$$\mathbb{E}_0^{\otimes n} [u(x_1, t)^p] \geq p_1^\rho(0, x_1 - x_2)^n \mathbb{E}_0^{\otimes n} ([u(x_2, t-1)]^p).$$

**Step 2: Variational representation.** From now on, we restrict our attention to the case  $x = 0$ . The aim of this step is to give a variational representation of  $\lambda_p^{(n)}(\kappa, \rho)$ . To this end, we introduce further notations. Let  $(e_1, \dots, e_d)$  be the canonical basis of  $\mathbb{R}^d$ . For  $x = (x_1, \dots, x_p) \in \mathbb{Z}^{dp}$ , and  $f : (x, y) \in \mathbb{Z}^{dp} \times \mathbb{Z}^{dn} \mapsto \mathbb{R}$ , we set

$$\nabla_x f(x, y) = (\nabla_{x_1} f(x, y), \dots, \nabla_{x_p} f(x, y)) \in \mathbb{R}^{dp},$$

where for  $j \in \{1, \dots, p\}$ , and  $i \in \{1, \dots, d\}$ ,

$$\langle \nabla_{x_j} f(x, y), e_i \rangle = f(x_1, \dots, x_j + e_i, \dots, x_p, y) - f(x, y).$$

The same notation is used for the  $y$ -coordinates, so that  $\nabla_y f(x, y) \in \mathbb{R}^{dn}$ . We also define

$$\begin{aligned} \Delta_x f(x, y) &= \sum_{j=1}^p \Delta_{x_j} f(x, y) \\ &= \sum_{j=1}^p \sum_{\substack{z_j \in \mathbb{Z}^d \\ z_j \sim x_j}} [f(x_1, \dots, z_j, \dots, x_p, y) - f(x_1, \dots, x_j, \dots, x_p, y)]. \end{aligned}$$

**Proposition 2.1.** *Let  $d \geq 1$  and  $n, p \in \mathbb{N}$ . For all  $\kappa, \rho \in [0, \infty)$ ,*

$$\begin{aligned} \lambda_p^{(n)}(\kappa, \rho) &= \lim_{t \rightarrow \infty} \frac{1}{pt} \log \mathbb{E}_0^{\otimes n} [u(0, t)^p] \\ &= \frac{1}{p} \sup_{\substack{f \in l^2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn}) \\ \|f\|_2 = 1}} \left\{ -\kappa \|\nabla_x f\|_2^2 - \rho \|\nabla_y f\|_2^2 + \sum_{(x, y)} I_p(x, y) f^2(x, y) \right\}. \end{aligned} \quad (20)$$

*Proof. Upper bound.* For a positive integer  $m$ , let  $B_R^m$  denote the ball in  $\mathbb{Z}^{dm}$  of radius  $R = t \log(t)$  centered at the origin. We first prove the following lemma which states we can restrict (18) to  $X$  paths being in  $B_R^p$  at time  $t$  and  $Y$  paths starting from  $B_R^n$ .

**Lemma 2.1.** *As  $t \rightarrow \infty$ ,*

$$\mathbb{E}_0^{\otimes n} [u(x, t)^p] = (1 + o(1)) \sum_{z \in B_R^n} \mathbb{E}_{0, z}^{X, Y} \left[ \exp \left( \int_0^t I_p(X_s, Y_s) ds \right) \delta_0(Y_t) \mathbf{1}_{(B_R^p)}(X_t) \right]. \quad (21)$$

*Proof.* It is enough to prove that

$$r(t) := \frac{\mathbb{E}_0^{\otimes n} [u(x, t)^p] - \sum_{z \in B_R^n} \mathbb{E}_{0, z}^{X, Y} \left[ \exp \left( \int_0^t I_p(X_s, Y_s) ds \right) \delta_0(Y_t) \mathbf{1}_{(B_R^p)}(X_t) \right]}{\mathbb{E}_0^{\otimes n} [u(x, t)^p]} \quad (22)$$

converges to 0 as  $t \rightarrow \infty$ . Using the trivial bounds

$$1 \leq \exp \left( \int_0^t I_p(X_s, Y_s) ds \right) \leq \exp(tnp) \quad (23)$$

and splitting the sum in (18), we get

$$\begin{aligned}
r(t) &\leq \frac{e^{tnp}}{\sum_{z \in \mathbb{Z}^{dn}} \mathbb{E}_{0,z}^{X,Y} [\delta_0(Y_t)]} \left( \sum_{z \notin B_R^n} \mathbb{E}_{0,z}^{X,Y} [\delta_0(Y_t)] + \sum_{z \in B_R^n} \mathbb{E}_{0,z}^{X,Y} [\delta_0(Y_t) \mathbf{1}_{(B_R^p)^c}(X_t)] \right) \\
&\leq \frac{e^{tnp}}{\sum_{z \in \mathbb{Z}^{dn}} \mathbb{P}_z(Y_t = 0)} \left( \sum_{z \notin B_R^n} \mathbb{P}_z(Y_t = 0) + \mathbb{P}_0(X_t \notin B_R^p) \sum_{z \in B_R^n} \mathbb{P}_z(Y_t = 0) \right) \\
&\leq e^{tnp} (\mathbb{P}_0(Y_t \notin B_R^n) + \mathbb{P}_0(X_t \notin B_R^p)),
\end{aligned}$$

where for the last two inequalities we used the time-reversal of  $Y$ . We have for  $R = t \log(t)$  and large enough  $t$

$$\begin{aligned}
\mathbb{P}_0(Y_1^p(t) \notin B_R^1) &\leq \exp[-C(d, \rho)t \log(t)], \\
\mathbb{P}_0(X_1^\kappa(t) \notin B_R^1) &\leq \exp[-C(d, \kappa)t \log(t)]
\end{aligned} \tag{24}$$

for some positive constants  $C(d, \rho)$  and  $C(d, \kappa)$  (see for instance Lemma 4.3 in [10]). Using this we get

$$r(t) \leq e^{tnp} \left( n e^{-C(d, \rho)t \log t} + p e^{-C(d, \kappa)t \log t} \right) \xrightarrow{t \rightarrow \infty} 0.$$

This finishes the proof of the lemma.  $\square$

Using Lemma 2.1 it is enough to study the existence of

$$\begin{aligned}
&\lim_{t \rightarrow \infty} \frac{1}{t} \log \sum_{z \in B_R^n} \mathbb{E}_{0,z}^{X,Y} \left[ \exp \left( \int_0^t I_p(X_s, Y_s) ds \right) \delta_0(Y_t) \mathbf{1}_{B_R^p}(X_t) \right] \\
&= \lim_{t \rightarrow \infty} \frac{1}{t} \log \langle f_1, e^{t\mathcal{L}_p} f_2 \rangle,
\end{aligned}$$

where  $f_1 : (x, y) \in \mathbb{Z}^{dp} \times \mathbb{Z}^{dn} \mapsto \delta_0(x) \mathbf{1}_{B_R^n}(y)$ ,  $f_2 : (x, y) \in \mathbb{Z}^{dp} \times \mathbb{Z}^{dn} \mapsto \mathbf{1}_{B_R^p}(x) \delta_0(y)$ , and  $\mathcal{L}_p$  is the bounded self-adjoint operator in  $l^2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn})$  defined by

$$\mathcal{L}_p f(x, y) = \kappa \Delta_x f(x, y) + \rho \Delta_y f(x, y) + I_p(x, y) f(x, y), \quad (x, y) \in \mathbb{Z}^{dp} \times \mathbb{Z}^{dn}.$$

For a linear operator  $\mathcal{L}$  on  $l^2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn})$  we define

$$\|\mathcal{L}\|_{2,2} := \sup_{\substack{f \in l^2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn}) \\ \|f\|_2 = 1}} \langle f, \mathcal{L} f \rangle.$$

Note that we have

$$\langle f_1, e^{t\mathcal{L}_p} f_2 \rangle \leq \|f_1\|_2 \left\| e^{t\mathcal{L}_p} \right\|_{2,2} \|f_2\|_2 = C(d, n, p) R^{d(n+p)/2} \left\| e^{t\mathcal{L}_p} \right\|_{2,2},$$

for some constant  $C(d, n, p) > 0$ . Thus,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \langle f_1, e^{t\mathcal{L}_p} f_2 \rangle \leq \|\mathcal{L}_p\|_{2,2} = \sup_{\substack{f \in l^2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn}) \\ \|f\|_2=1}} \langle f, \mathcal{L}_p f \rangle,$$

which is the upper bound in (20).

**Lower bound.** By (18) with  $x = 0$ , it follows that

$$\begin{aligned} \mathbb{E}_0^{\otimes n} [u(0, t)^p] &\geq \mathbb{E}_{0,0}^{X,Y} \left[ \exp \left( \int_0^t I_p(X_s, Y_s) ds \right) \delta_0(X_t) \delta_0(Y_t) \right] \\ &= \langle \delta_0 \otimes \delta_0, e^{t\mathcal{L}_p}(\delta_0 \otimes \delta_0) \rangle = \left\| e^{\frac{t}{2}\mathcal{L}_p}(\delta_0 \otimes \delta_0) \right\|_2^2 \\ &= \sum_{x \in \mathbb{Z}^{dp}} \sum_{y \in \mathbb{Z}^{dn}} \left( e^{\frac{t}{2}\mathcal{L}_p}(\delta_0 \otimes \delta_0)(x, y) \right)^2. \end{aligned}$$

Restricting the sum over  $B_R^p \times B_R^n$ , and applying Jensen's inequality, we get

$$\begin{aligned} \mathbb{E}_0^{\otimes n} [u(0, t)^p] &\geq \sum_{x \in B_R^p} \sum_{y \in B_R^n} \left( e^{\frac{t}{2}\mathcal{L}_p}(\delta_0 \otimes \delta_0)(x, y) \right)^2 \\ &\geq \frac{1}{|B_R^n|} \frac{1}{|B_R^p|} \left( \sum_{x \in B_R^p} \sum_{y \in B_R^n} e^{\frac{t}{2}\mathcal{L}_p}(\delta_0 \otimes \delta_0)(x, y) \right)^2 \\ &= \frac{C(d, n, p)}{R^{d(n+p)}} \left( \sum_{x \in B_R^p} \sum_{y \in B_R^n} \mathbb{E}_{x,y}^{X,Y} \left[ \exp \left( \int_0^{t/2} I_p(X_s, Y_s) ds \right) \delta_0(X_{t/2}) \delta_0(Y_{t/2}) \right] \right)^2 \\ &= \frac{C(d, n, p)}{R^{d(n+p)}} \left( \mathbb{E}_{0,0}^{X,Y} \left[ \exp \left( \int_0^{t/2} I_p(X_s, Y_s) ds \right) \mathbf{1}_{B_R^p}(X_{t/2}) \mathbf{1}_{B_R^n}(Y_{t/2}) \right] \right)^2. \end{aligned}$$

Taking  $R = t \log(t)$ , we obtain that

$$\begin{aligned} \liminf_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_0^{\otimes n} [u(0, t)^p] &\geq \liminf_{t \rightarrow \infty} \frac{2}{t} \log \mathbb{E}_{0,0}^{X,Y} \left[ \exp \left( \int_0^{t/2} I_p(X_s, Y_s) ds \right) \mathbf{1}_{B_R^p}(X_{t/2}) \mathbf{1}_{B_R^n}(Y_{t/2}) \right]. \end{aligned}$$

On the other hand, by (23), (24) and our choice of  $R$ , we have

$$\begin{aligned} \mathbb{E}_{0,0}^{X,Y} \left[ \exp \left( \int_0^{t/2} I_p(X_s, Y_s) ds \right) \mathbf{1}_{(B_R^p \times B_R^n)^c}(X_{t/2}, Y_{t/2}) \right] &\leq \exp \left( \frac{tnp}{2} \right) \mathbb{P}_0(X_{t/2} \notin B_R^p) \mathbb{P}_0(Y_{t/2} \notin B_R^n) \\ &\leq np \exp \left[ \frac{tnp}{2} - (C(d, \rho) + C(d, \kappa)) t \log(t) \right], \end{aligned}$$

and therefore, with a similary reasoning as in the proof of Lemma 2.1 we get

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_0^{\otimes n} [u(0, t)^p] \geq \liminf_{t \rightarrow \infty} \frac{2}{t} \log \mathbb{E}_{0,0}^{X,Y} \left[ \exp \left( \int_0^{t/2} I_p(X_s, Y_s) ds \right) \right].$$

Now, the occupation measure  $\frac{1}{t} \int_0^t \delta_{(X_s, Y_s)} ds$  satisfies a weak large deviations principle (LDP) in the space  $\mathcal{M}_1(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn})$  of probability measures on  $\mathbb{Z}^{dp} \times \mathbb{Z}^{dn}$ , endowed with the weak topology. The speed of this LDP is  $t$  and the rate function is given for all  $\nu \in \mathcal{M}_1(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn})$  by

$$J(\nu) = \kappa \|\nabla_x \sqrt{\nu}\|_2^2 + \rho \|\nabla_y \sqrt{\nu}\|_2^2,$$

(see e.g. den Hollander [12], Section IV.4). Since  $I$  is bounded, the lower bound in Varadhan's integral lemma (see e.g. den Hollander [12], Section III.3) yields

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}_0^{\otimes n} [u(0, t)^p] \geq \sup_{\nu \in \mathcal{M}_1(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn})} \left\{ \sum_{(x,y)} I_p(x, y) \nu(x, y) - J(\nu) \right\}.$$

Setting  $f(x, y) = \sqrt{\nu}(x, y)$  gives then the lower bound in (20).  $\square$

**Step 3: Properties of  $\lambda_p^{(n)}$ .** Since  $0 \leq I_p(x, y) \leq np$ , we clearly have  $0 \leq \lambda_p^{(n)} \leq n$ . Using representation (20), we can conclude that the function  $(\kappa, \rho) \mapsto \lambda_p^{(n)}(\kappa, \rho)$  is convex and nonincreasing in  $\kappa$  and  $\rho$ . Moreover,  $\lambda_p^{(n)}(\kappa, \rho)$  is lower semi-continuous since it is supremum of functions that are linear in  $\kappa$  and  $\rho$ . Finally, since every finite convex function is also upper semi-continuous,  $\lambda_p^{(n)}$  is upper semi-continuous. Hence,  $\lambda_p^{(n)}(\kappa, \rho)$  is continuous.

### 3 Proof of Theorems 1.2–1.3

By symmetry, note that for all  $n, p \in \mathbb{N}$  and  $\kappa, \rho \in [0, \infty)$ ,

$$\lambda_p^{(n)}(\kappa, \rho) = \frac{n}{p} \lambda_n^{(p)}(\rho, \kappa). \quad (25)$$

#### 3.1 Proof of Theorem 1.2

**Proof of (i):** By continuity,  $\lim_{\kappa \rightarrow 0} \lambda_p^{(n)}(\kappa, \rho) = \lambda_p^{(n)}(0, \rho)$ . Now for  $\kappa = 0$ , the  $X$  particles do not move so that  $\mathbb{E}_0^{\otimes n} [u(0, t)^p] = \mathbb{E}_0 \left( \exp(p L_t^Y(0)) \right)^n$  (see (19)), where  $L_t^Y(0)$  is the local time at 0 of a simple random walk in  $\mathbb{Z}^d$  with rate  $2d\rho$ . Using the LDP for  $L_t^Y$ , we obtain



$$\lambda_p^{(n)}(0, \rho) = \frac{n}{p} \sup_{\substack{f \in l^2(\mathbb{Z}^d) \\ \|f\|_2=1}} \langle f, (\rho \Delta + p \delta_0) f \rangle = n\mu(\rho/p).$$

**Proof of (ii):** For all  $n, p \in \mathbb{N}$  and  $\kappa, \rho \in [0, \infty)$ , we have

$$\lambda_p^{(n)}(\kappa, \rho) \geq \lambda_1^{(n)}(\kappa, \rho) = n\lambda_n^{(1)}(\rho, \kappa) \geq n\lambda_1^{(1)}(\rho, \kappa) = n\mu(\kappa + \rho),$$

where the last equality is proved in [3] and comes from the fact that  $X_t^1 - Y_t^1$  is a simple random walk in  $\mathbb{Z}^d$  with jump rate  $2d(\kappa + \rho)$ . Since  $G_d(0) = \infty$  for  $d = 1, 2$ , it follows from (9) that  $\lambda_p^{(n)}(\kappa, \rho) > 0$  for  $d = 1, 2$ .

Let us prove that  $\lim_{\kappa \rightarrow \infty} \lambda_p^{(n)}(\kappa, \rho) = 0$ . By monotonicity in  $\rho$ ,

$$\lambda_p^{(n)}(\kappa, \rho) \leq \lambda_p^{(n)}(\kappa, 0) = n\mu(\kappa/n). \quad (26)$$

Hence the only thing to prove is that  $\lim_{\kappa \rightarrow \infty} \mu(\kappa) = 0$ . To this end, one can use the discrete Gagliardo-Nirenberg inequality: there exists a constant  $C$  such that for all  $f : \mathbb{Z}^d \mapsto \mathbb{R}$ ,

$$\text{for } d = 1, \quad \|f\|_\infty^2 \leq C \|f\|_2 \|\nabla f\|_2; \quad (27)$$

$$\text{for } d = 2, \quad \|f\|_4^2 \leq C \|f\|_2 \|\nabla f\|_2. \quad (28)$$

The proof of this inequality follows the same lines as the proof of the usual Gagliardo-Nirenberg inequality (see Brezis [1]). Since we could not find a reference, we give a short proof of (27) and (28) in the appendix. From (27) and (28), we get for all  $f \in l_2(\mathbb{Z}^d)$  with  $\|f\|_2 = 1$ ,

$$\begin{aligned} -\kappa \|\nabla f\|_2^2 + f(0)^2 &\leq \begin{cases} -\kappa \|\nabla f\|_2^2 + \|f\|_\infty^2 & \text{for } d = 1 \\ -\kappa \|\nabla f\|_2^2 + \|f\|_4^2 & \text{for } d = 2 \end{cases} \\ &\leq -\kappa \|\nabla f\|_2^2 + C \|\nabla f\|_2. \end{aligned}$$

Taking the supremum over  $f$  yields

$$\mu(\kappa) \leq \sup_{x \geq 0} (-\kappa x^2 + Cx) = \frac{C^2}{4\kappa}.$$

The strict monotonicity is now an easy consequence of the fact that  $\kappa \mapsto \lambda_p^{(n)}(\kappa, \rho)$  is convex, positive, non increasing, and tends to 0 as  $\kappa \rightarrow \infty$ .

**Proof of (iii):** By (25) and (26), we get

$$\lambda_p^{(n)}(\kappa, \rho) \leq n \min(\mu(\kappa/n), \mu(\rho/p)). \quad (29)$$

Then the claim follows by (9).

### 3.2 Proof of Theorem 1.3

**Proof of (i):** Fix  $\varepsilon > 0$ . Let  $f$  approaching the supremum in the variational representation (20) of  $\lambda_p^{(n)}(\kappa, 0)$ , so that

$$\begin{aligned} p\lambda_p^{(n)}(\kappa, 0) - \varepsilon &\leq -\kappa \|\nabla_x f\|_2^2 + \sum_{x \in \mathbb{Z}^{dp}} \sum_{y \in \mathbb{Z}^{dn}} I_p(x, y) f^2(x, y) \\ &\leq p\lambda_p^{(n)}(\kappa, \rho) + \rho \sup_{\substack{f \in l^2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn}) \\ \|f\|_2 = 1}} \|\nabla_y f\|_2^2. \end{aligned}$$

For  $x \in \mathbb{Z}^{dp}$ , set  $f_x : y \in \mathbb{Z}^{dn} \mapsto f(x, y)$ . Since the bottom of the spectrum of  $\Delta$  in  $l^2(\mathbb{Z}^{dn})$  is  $-4dn$ ,

$$\sum_{y \in \mathbb{Z}^{dn}} \|\nabla_y f_x(y)\|_2^2 \leq 4dn \sum_{y \in \mathbb{Z}^{dn}} f_x^2(y),$$

for all  $x \in \mathbb{Z}^{dp}$ . Hence,

$$\sum_{x \in \mathbb{Z}^{dp}} \sum_{y \in \mathbb{Z}^{dn}} \|\nabla_y f_x(y)\|_2^2 \leq 4dn \sum_{x \in \mathbb{Z}^{dp}} \sum_{y \in \mathbb{Z}^{dn}} f_x^2(y) = 4dn.$$

Therefore, for all  $\varepsilon > 0$ ,

$$p\lambda_p^{(n)}(\kappa, 0) - \varepsilon \leq p\lambda_p^{(n)}(\kappa, \rho) + 4dn\rho.$$

Letting  $\varepsilon \rightarrow 0$  yields,

$$\lambda_p^{(n)}(\kappa, 0) - \frac{4dn\rho}{p} \leq \lambda_p^{(n)}(\kappa, \rho) \leq \lambda_p^{(n)}(\kappa, 0), \quad (30)$$

which, after letting  $p \rightarrow \infty$ , gives the claim.

**Proof of (ii):** By (25),  $\lim_{n \rightarrow \infty} \lambda_p^{(n)}(\kappa, \rho) = \lim_{n \rightarrow \infty} \frac{n}{p} \lambda_n^{(p)}(\rho, \kappa)$  and by (i),

$$\lim_{n \rightarrow \infty} \lambda_n^{(p)}(\rho, \kappa) \geq \lambda_n^{(p)}(\rho, 0) = p\mu(\rho/p) > 0, \text{ for } p > \rho/G_d(0).$$

Hence, for  $p > \rho/G_d(0)$ ,  $\lim_{n \rightarrow \infty} \lambda_p^{(n)}(\kappa, \rho) = +\infty$ .

**Proof of (iii):** This is a direct consequence of Theorem 1.2(iii).

## 4 Proof of Theorem 1.4

**Proof of (i):** We first prove that

$$\kappa_p^{(n)}(\rho) = \sup_{\substack{f \in l_2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn}) \\ \|f\|_2=1}} \frac{\sum_{x,y} I_p(x,y) f^2(x,y) - \rho \|\nabla_y f\|_2^2}{\|\nabla_x f\|_2^2}, \quad (31)$$

with  $I$  defined as in (17). Indeed, let us denote by  $S$  the supremum in the right-hand side of (31).

If  $\kappa \geq \kappa_p^{(n)}(\rho)$ , then  $\lambda_p^{(n)}(\kappa, \rho) = 0$ . Therefore, using (20), for all  $f \in l_2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn})$  such that  $\|f\|_2 = 1$ ,

$$\sum_{x \in \mathbb{Z}^{dp}} \sum_{y \in \mathbb{Z}^{dn}} I_p(x,y) f^2(x,y) - \rho \|\nabla_y f\|_2^2 \leq \kappa \|\nabla_x f\|_2^2,$$

so that  $\kappa \geq S$ . Hence  $\kappa_p^{(n)}(\rho) \geq S$ . On the opposite direction, we can assume that  $S < \infty$ . Then, by definition of  $S$ , for all  $f \in l_2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn})$  such that  $\|f\|_2 = 1$ ,

$$\sum_{x \in \mathbb{Z}^{dp}} \sum_{y \in \mathbb{Z}^{dn}} I_p(x,y) f^2(x,y) - \rho \|\nabla_y f\|_2^2 \leq S \|\nabla_x f\|_2^2.$$

Thus, for all  $f \in l_2(\mathbb{Z}^{dp} \times \mathbb{Z}^{dn})$  such that  $\|f\|_2 = 1$ , and all  $\kappa \geq S$ ,

$$\sum_{x \in \mathbb{Z}^{dp}} \sum_{y \in \mathbb{Z}^{dn}} I_p(x,y) f^2(x,y) - \rho \|\nabla_y f\|_2^2 - \kappa \|\nabla_x f\|_2^2 \leq (S - \kappa) \|\nabla_x f\|_2^2 \leq 0.$$

Hence, for all  $\kappa \geq S$ ,  $\lambda_p^{(n)}(\kappa, \rho) = 0$ , i.e.,  $\kappa \geq \kappa_p^{(n)}(\rho)$ . Hence,  $S \geq \kappa_p^{(n)}(\rho)$ . This proves (31).

Since  $\rho \mapsto \kappa_p^{(n)}(\rho)$  is a supremum of linear functions, it is lower semi-continuous and convex. It is also obvious that  $\rho \mapsto \kappa_p^{(n)}(\rho)$  is non increasing. The continuity follows then from the finiteness of  $\kappa_p^{(n)}(\rho)$ .

The lower bound in (13) is a direct consequence of (30). Indeed, since  $\lambda_p^{(n)}(\kappa, 0) = n\mu(\kappa/n)$ , it follows from (30) that if  $\mu(\kappa/n) > 4d\rho/p$ , then  $\kappa < \kappa_p^{(n)}(\rho)$ . This yields the bound:

$$\kappa_p^{(n)}(\rho) \geq n\mu^{-1}(4d\rho/p).$$

Using the symmetry relation (25), we also get from (30) that

$$\lambda_p^{(n)}(\kappa, \rho) \geq n\mu(\rho/p) - 4d\kappa.$$

This leads to  $\kappa_p^{(n)}(\rho) \geq \frac{n}{4d}\mu(\rho/p)$ . Hence, if  $\rho/p < G_d(0)$ ,  $\kappa_p^{(n)}(\rho) > 0$ . We have already seen that  $\kappa_p^{(n)}(\rho) = 0$  if  $\rho/p \geq G_d(0)$ . Since  $\lambda_p^{(n)}(\kappa, 0) = n\mu(\kappa/n)$ , it follows that  $\kappa_p^{(n)}(0) = nG_d(0)$ . Using convexity, we have, for all  $\rho \in [0, pG_d(0)]$ ,

$$\kappa_p^{(n)}(\rho) \leq \frac{\kappa_p^{(n)}(pG_d(0)) - \kappa_p^{(n)}(0)}{pG_d(0)} \rho + \kappa_p^{(n)}(0) = n(G_d(0) - \rho/p).$$

Since  $\kappa_p^{(n)}(\rho) = 0$  if  $\rho/p \geq G_d(0)$ , then the upper bound in (13) is proved.

**Proof of (ii):** To prove (14), let  $f_0$  be the function

$$f_0(x, y) = \prod_{i=1}^p \frac{G_d(x_i)}{\|G_d\|_2} \prod_{j=1}^n \delta_0(y_j).$$

Note that for  $d \geq 5$ ,  $\|G_d\|_2 < \infty$ , so that  $f_0$  is well-defined, and has  $l_2$ -norm equal to 1. From (31), we get

$$\kappa_p^{(n)}(\rho) \geq \frac{\sum_{x,y} I_p(x, y) f_0^2(x, y) - \rho \|\nabla_y f_0\|_2^2}{\|\nabla_x f_0\|_2^2}.$$

An easy computation then gives

$$\sum_{x,y} I_p(x, y) f_0^2(x, y) = np \frac{G_d^2(0)}{\|G_d\|_2^2},$$

$$\|\nabla_y f_0\|_2^2 = n \|\nabla_{y_1} \delta_0\|_2^2 = 2dn,$$

and

$$\|\nabla_x f_0\|_2^2 = p \frac{\|\nabla_{x_1} G_d\|_2^2}{\|G_d\|_2^2} = p \frac{G_d(0)}{\|G_d\|_2^2},$$

since  $\|\nabla_{x_1} G_d\|_2^2 = \langle G_d, -\Delta G_d \rangle = \langle G_d, \delta_0 \rangle = G_d(0)$ . This gives (14).

**Proof of (iii):** The inequality (15) is clear if  $\rho \in [(p-1)G_d(0), pG_d(0)]$ , since in this case,  $\kappa_{p-1}^{(n)}(\rho) = 0 < \kappa_p^{(n)}(\rho)$ . We assume therefore that  $\rho \in (0, (p-1)G_d(0))$ . From (13), we have  $\kappa_{p-1}^{(n)}(\rho) \leq nG_d(0) - \rho n/(p-1)$ , whereas, from (14),  $\kappa_p^{(n)}(\rho) \geq nG_d(0) - \rho n/(p\alpha_d)$ . Hence  $\kappa_{p-1}^{(n)}(\rho) < \kappa_p^{(n)}(\rho)$  as soon as  $\alpha_d > \frac{p-1}{p}$ . This gives the claim.

## 5 Proof of Corollary 1.1

**Proof of (i):** The function  $p \mapsto \lambda_p^{(n)}(\kappa, \rho)$  increases from  $\lambda_1^{(n)}(\kappa, \rho)$  to  $n\mu(\kappa/n)$ . Hence, there exists  $p$  such that  $\lambda_p^{(n)}(\kappa, \rho) < \lambda_{p+1}^{(n)}(\kappa, \rho)$  as soon as  $\lambda_1^{(n)}(\kappa, \rho) < n\mu(\kappa/n)$ . But  $n\mu(\kappa/n) = \lambda_1^{(n)}(\kappa, 0)$ . Hence, if  $\lambda_1^{(n)}(\kappa, \rho) = n\mu(\kappa/n)$ , the convex decreasing function  $\rho \mapsto \lambda_1^{(n)}(\kappa, \rho)$  is constant. Being equal to 0 for  $\rho \geq G_d(0)$ , we get that  $n\mu(\kappa/n) = 0$ , which can not be the case if  $\kappa < nG_d(0)$ . This ends the proof of the first part.

If  $\kappa \geq nG_d(0)$ , then  $\lambda_p^{(n)}(\kappa, \rho) = 0$ , for all  $p \geq 1$ , and the system is not intermittent. This proves the second part.

**Proof of (ii):** For all  $p \in \mathbb{N} \setminus \{1\}$  by Lemma 1.1 for  $d$  large enough we have  $\alpha_d > \frac{p-1}{p}$ . This implies that  $\alpha_d > \frac{q-1}{q}$  for all  $q \in \mathbb{N} \setminus \{1\}$  and  $q \leq p$ . Hence, by Theorem 1.4(iii), for all  $q \in \mathbb{N} \setminus \{1\}$  with  $q \leq p$  we have  $\kappa_{q-1}^{(n)}(\rho) < \kappa_q^{(n)}(\rho)$ , for all  $\rho \in (0, pG_d(0))$ . Hence, in the domain

$$\left\{ (\kappa, \rho) : \rho \in (0, qG_d(0)), \kappa_{q-1}^{(n)}(\rho) \leq \kappa < \kappa_q^{(n)}(\rho) \right\}$$

one has

$$\lambda_1^{(n)}(\kappa, \rho) = \dots = \lambda_{q-1}^{(n)}(\kappa, \rho) = 0 < \lambda_q^{(n)}(\kappa, \rho),$$

which proves the desired result.

**Acknowledgements** The research in this paper was supported by the ANR-project MEMEMO.

## Appendix: Proof of lemma 1.1

For a function  $f : \mathbb{Z}^d \mapsto \mathbb{R}$ , let  $\hat{f}$  denote the Fourier transform of  $f$ :

$$\hat{f}(\theta) = \sum_{x \in \mathbb{Z}^d} e^{i\langle \theta, x \rangle} f(x) \quad \forall \theta \in [0, 2\pi]^d.$$

Then, the inverse Fourier transform is given by

$$f(x) = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} e^{-i\langle \theta, x \rangle} \hat{f}(\theta) d\theta,$$

and the Plancherel's formula reads

$$\sum_{x \in \mathbb{Z}^d} f^2(x) = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} |\hat{f}(\theta)|^2 d\theta.$$

Using the equation  $\Delta G_d = -\delta_0$  we get that

$$\hat{G}_d(\theta) = \frac{1}{2 \sum_{i=1}^d (1 - \cos(\theta_i))}.$$

Hence,

$$\begin{aligned}
G_d(0) &= \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} \frac{d\theta}{2 \sum_{i=1}^d (1 - \cos(\theta_i))} \\
&= \frac{1}{\pi^d} \int_{[0, \pi]^d} \frac{d\theta}{2 \sum_{i=1}^d (1 - \cos(\theta_i))} \\
&= \mathbb{E} \left[ \frac{1}{2 \sum_{i=1}^d (1 - \cos(\Theta_i))} \right]
\end{aligned}$$

where the random variables  $(\Theta_i)$  are i.i.d. with uniform distribution on  $[0, \pi]$ . Moreover, by Plancherel's formula we have

$$\|G_d\|_2^2 = \frac{1}{(2\pi)^d} \int_{[0, 2\pi]^d} \frac{d\theta}{(2 \sum_{i=1}^d (1 - \cos(\theta_i)))^2} = \mathbb{E} \left[ \frac{1}{(2 \sum_{i=1}^d (1 - \cos(\Theta_i)))^2} \right].$$

Thus,

$$\alpha_d = \frac{G_d(0)}{2d \|G_d\|_2^2} = \frac{\mathbb{E} \left[ \frac{1}{\bar{S}_d} \right]}{\mathbb{E} \left[ \frac{1}{\bar{S}_d^2} \right]},$$

where  $\bar{S}_d = \frac{1}{d} \sum_{i=1}^d (1 - \cos(\Theta_i))$ . Applying Hölder's and Jensen's inequality, we get that

$$\alpha_d \leq \frac{1}{\sqrt{\mathbb{E} \left[ \frac{1}{\bar{S}_d^2} \right]}} \leq \mathbb{E}(\bar{S}_d) = 1.$$

By the law of large numbers,  $\bar{S}_d$  converges almost surely to  $\mathbb{E}[1 - \cos(\Theta)] = 1$  as  $d$  tends to infinity. We are now going to prove that  $\bar{S}_d^{-2}$  is uniformly integrable by showing that for all  $p > 2$ ,

$$\sup_{d > 2p} \mathbb{E} \left[ \bar{S}_d^{-p} \right] < \infty. \quad (32)$$

Indeed, let  $\varepsilon \in (0, \pi)$  be a small positive number to be fixed later. Let

$$\mathcal{J} = \{i \in \{1, \dots, d\} : 0 \leq \Theta_i \leq \varepsilon\}.$$

$$\bar{S}_d \geq \frac{1}{d} \sum_{i \notin \mathcal{J}} (1 - \cos(\varepsilon)) + \frac{c_\varepsilon}{d} \sum_{i \in \mathcal{J}} \Theta_i^2,$$

where  $c_\varepsilon = \inf_{0 \leq \theta \leq \varepsilon} \frac{1 - \cos(\theta)}{\theta^2} \rightarrow 1/2$  when  $\varepsilon \rightarrow 0$ . Therefore,

$$\mathbb{E} \left[ \bar{S}_d^{-p} \right] \leq d^p \sum_{k=0}^d \sum_{\substack{I \subset \{1, \dots, d\} \\ |I|=k}} \mathbb{E} \left[ \frac{\mathbf{1}_{\mathcal{J}=I}}{((1 - \cos(\varepsilon))(d - k) + c_\varepsilon \sum_{i \in I} \Theta_i^2)^p} \right].$$

Since the last expectation only depends on  $|I|$ , we get

$$\mathbb{E} \left[ \bar{S}_d^{-p} \right] \leq d^p \sum_{k=0}^d \binom{d}{k} a(k, \varepsilon, d),$$

with

$$a(k, \varepsilon, d) := \frac{1}{\pi^d} \int_{\substack{0 \leq \theta_1, \dots, \theta_k \leq \varepsilon \\ \varepsilon \leq \theta_{k+1}, \dots, \theta_d \leq \pi}} \frac{d\theta_1 \cdots d\theta_d}{((1 - \cos(\varepsilon))(d - k) + c_\varepsilon(\theta_1^2 + \cdots + \theta_k^2))^p}.$$

Let  $\omega_d$  denote the volume of the  $d$ -dimensional unit ball. For  $k = d$ ,

$$\begin{aligned} a(d, \varepsilon, d) &= \frac{1}{\pi^d} \int_{0 \leq \theta_1, \dots, \theta_d \leq \varepsilon} \frac{d\theta_1 \cdots d\theta_d}{c_\varepsilon^p \|\theta\|^{2p}} \\ &\leq \frac{1}{c_\varepsilon^p \pi^d} \omega_d \int_0^{\sqrt{d}\varepsilon} r^{d-2p-1} dr \\ &= \left(\frac{\varepsilon}{\pi}\right)^d \frac{1}{(c_\varepsilon \varepsilon^2)^p} d^{\frac{d}{2}-p} \frac{\omega_d}{d-2p}, \end{aligned}$$

for  $d > 2p$ .

Note that for large  $d$ ,  $\omega_d \simeq \frac{(2e\pi)^{d/2}}{\sqrt{\pi d} d^{d/2}}$ . Therefore, as  $d \rightarrow \infty$

$$d^p \binom{d}{d} a(d, \varepsilon, d) = O\left(d^{-3/2} (\varepsilon^2 2e/\pi)^{d/2}\right).$$

If  $\varepsilon$  is chosen so that  $\varepsilon^2 \leq \pi/(2e)$ , we obtain that  $\lim_{d \rightarrow \infty} d^p \binom{d}{d} a(d, \varepsilon, d) = 0$ .

For  $k \leq d-1$ ,

$$a(k, \varepsilon, d) \leq \frac{1}{(1 - \cos(\varepsilon))^p} \frac{1}{(d-k)^p} \left(\frac{\varepsilon}{\pi}\right)^k \left(1 - \frac{\varepsilon}{\pi}\right)^{d-k},$$

and  $d^p \binom{d}{k} a(k, \varepsilon, d) \leq \frac{1}{(1 - \cos(\varepsilon))^p} \mathbb{E}[\mathbf{1}_{N=k} (1 - N/d)^{-p}]$ , where  $N$  is a Binomial random variable with parameters  $d$  and  $\varepsilon/\pi$ . Hence, for  $\varepsilon < \min(\pi, \sqrt{\pi/(2e)})$ ,

$$\begin{aligned} &\mathbb{E} \left[ \frac{1}{\bar{S}_d^p} \right] \\ &\leq \frac{1}{(1 - \cos(\varepsilon))^p} \mathbb{E}[\mathbf{1}_{N \leq d-1} (1 - N/d)^{-p}] + O\left(d^{-3/2}\right) \\ &\leq \frac{d^p}{(1 - \cos(\varepsilon))^p} \mathbb{P} \left[ d \frac{2\varepsilon}{\pi} \leq N \leq d-1 \right] + \frac{1}{(1 - \cos(\varepsilon))^p (1 - \frac{2\varepsilon}{\pi})^p} + O\left(d^{-3/2}\right). \end{aligned}$$

Now, by the large deviations principle satisfied by  $N/d$ , there is an  $i(\varepsilon) > 0$  such that  $\mathbb{P}[N \geq d2\varepsilon/\pi] \leq \exp(-di(\varepsilon))$ . This ends the proof of (32).

Using the uniform integrability (32), and the fact that  $\bar{S}_d$  converges a.s. to 1, we obtain that  $\mathbb{E} \left[ \frac{1}{\bar{S}_d} \right]$  and  $\mathbb{E} \left[ \frac{1}{\bar{S}_d^2} \right]$  both converge to 1, when  $d$  goes to infinity.

### Appendix: Proof of proposition 1.1.

Let  $f(x, y_1, \dots, y_n)$  be a function in  $l_2(\mathbb{Z}^{d(1+n)})$  with  $\|f\|_2 = 1$ , such that  $\mathcal{L}_1 f = \lambda_1^{(n)}(\kappa, \rho)f$ . Set  $\tilde{f}(x_1, x_2, y_1, \dots, y_n) = f(x_1, y_1, \dots, y_n)f(x_2, y_1, \dots, y_n)$  and  $y = (y_1, \dots, y_n)$ . Note that

$$\sum_{x_1, x_2, y} \tilde{f}^2(x_1, x_2, y) = \sum_y \left( \sum_x f^2(x, y) \right)^2 \leq \left( \sup_y \sum_x f^2(x, y) \right) \|f\|_2^2 \leq \|f\|_2^4.$$

Hence,  $\tilde{f}$  is in  $l_2(\mathbb{Z}^{d(2+n)})$ . A simple computation yields

$$\triangle_{x_1} \tilde{f}(x_1, x_2, y) = f(x_2, y) \triangle_x f(x_1, y) \quad , \quad \triangle_{x_2} \tilde{f}(x_1, x_2, y) = f(x_1, y) \triangle_x f(x_2, y),$$

$$\begin{aligned} \triangle_y \tilde{f}(x_1, x_2, y) &= f(x_2, y) \triangle_y f(x_1, y) + f(x_1, y) \triangle_y f(x_2, y) \\ &\quad + \sum_{z \sim y} (f(x_1, z) - f(x_1, y))(f(x_2, z) - f(x_2, y)). \end{aligned}$$

Since  $l_2(x_1, x_2, y) = l_1(x_1, y) + l_1(x_2, y)$ , this leads to

$$\mathcal{L}_2 \tilde{f}(x_1, x_2, y) = 2\lambda_1^{(n)}(\kappa, \rho) \tilde{f}(x_1, x_2, y) + \rho \sum_{z \sim y} (f(x_1, z) - f(x_1, y))(f(x_2, z) - f(x_2, y)).$$

Therefore

$$\begin{aligned} \lambda_2^{(n)}(\kappa, \rho) \|\tilde{f}\|_2^2 &\geq \frac{1}{2} \langle \tilde{f}, \mathcal{L}_2 \tilde{f} \rangle \\ &= \lambda_1^{(n)}(\kappa, \rho) \|\tilde{f}\|_2^2 + \frac{\rho}{2} \sum_{y, z \sim y} \left( \sum_x f(x, y)(f(x, z) - f(x, y)) \right)^2 \end{aligned}$$

$\sum_{y, z \sim y} (\sum_x f(x, y)(f(x, z) - f(x, y)))^2 \geq 0$  and is equal to 0 if and only if for all  $y$  and  $z \sim y$ ,  $\sum_x f(x, y)(f(x, z) - f(x, y)) = 0$ . Inverting the role of  $z$  and  $y$  yields  $\sum_x (f(x, z) - f(x, y))^2 = 0$ , so that for all  $x$ , for all  $y$  and  $z \sim y$ ,  $f(x, z) = f(x, y)$ . Hence, for all  $x, y$ ,  $f(x, y) = f(x, 0)$ . This is impossible since  $\|f\|_2 = 1$ . Thus,  $\lambda_2^{(n)}(\kappa, \rho) > \lambda_1^{(n)}(\kappa, \rho)$ .  $\square$

### Appendix: Proof of the discrete Gagliardo Nirenberg inequality.

**Proof for  $d = 1$ .** One can assume that  $\|f\|_2 < \infty$ . Otherwise there is nothing to prove. Since  $\|f\|_2 < \infty$  we have  $\lim_{|x| \rightarrow \infty} |f(x)| = 0$ , and for all  $x \in \mathbb{Z}$ ,



$$\begin{aligned}
f^2(x) &= \sum_{j=-\infty}^x f^2(j) - f^2(j-1) \\
&\leq \sum_{j=-\infty}^{+\infty} |f(j) - f(j-1)| (|f(j)| + |f(j-1)|) \\
&\leq 2 \sqrt{\sum_j |f(j) - f(j-1)|^2} \sqrt{\sum_j f^2(j)} \\
&= 2 \|f\|_2 \|\nabla f\|_2.
\end{aligned}$$

This is (27) with a constant equal to 2.

**Proof for  $d = 2$ .** Here again, one can assume that  $\|f\|_2 < \infty$ , so that  $\lim_{|x_1| \rightarrow \infty} |f(x_1, x_2)| = 0$ . For all  $x_1, x_2 \in \mathbb{Z}$ ,

$$\begin{aligned}
f^2(x_1, x_2) &= \sum_{j_1=-\infty}^{x_1} f^2(j_1, x_2) - f^2(j_1 - 1, x_2) \\
&\leq \sum_{j_1=-\infty}^{+\infty} |f(j_1, x_2) - f(j_1 - 1, x_2)| (|f(j_1, x_2)| + |f(j_1 - 1, x_2)|) \\
&\leq 2 \sqrt{\sum_{j_1} |f(j_1, x_2) - f(j_1 - 1, x_2)|^2} \sqrt{\sum_{j_1} f^2(j_1, x_2)} \\
&= 2 \|f(\bullet, x_2)\|_2 \sqrt{\sum_{j_1} |\nabla_{x_1} f(j_1, x_2)|^2} := 2 \tilde{f}_1(x_2)
\end{aligned}$$

Similarly we have

$$f^2(x_1, x_2) \leq 2 \|f(x_1, \bullet)\|_2 \sqrt{\sum_{j_2} |\nabla_{x_2} f(x_1, j_2)|^2} := 2 \tilde{f}_2(x_1).$$

Therefore,

$$\sum_{x_1, x_2} f^4(x_1, x_2) \leq 4 \left( \sum_{x_2} \tilde{f}_1(x_2) \right) \left( \sum_{x_1} \tilde{f}_2(x_1) \right).$$

But,

$$\begin{aligned}
\sum_{x_2} \tilde{f}_1(x_2) &= \sum_{x_2} \|f(\bullet, x_2)\|_2 \sqrt{\sum_{j_1} |\nabla_{x_1} f(j_1, x_2)|^2} \\
&\leq \sqrt{\sum_{x_2} \|f(\bullet, x_2)\|_2^2} \sqrt{\sum_{x_2} \sum_{j_1} |\nabla_{x_1} f(j_1, x_2)|^2} \\
&\leq \|f\|_2 \|\nabla f\|_2.
\end{aligned}$$

The same is true for  $\sum_{x_1} \tilde{f}_2(x_1)$ , so that

$$\|f\|_4^4 \leq 4 \|f\|_2^2 \|\nabla f\|_2^2.$$

This proves (28) with a constant equal to 2.

## References

1. Brezis, H.: *Analyse fonctionnelle: Théorie et applications*. Collection Mat'ematiques Appliquées pour la Maîtrise, Masson, Paris (1983)
2. Carmona R.A., Molchanov S.A.: *Parabolic Anderson Problem and Intermittency*. AMS Memoir 518, American Mathematical Society, Providence RI (1994)
3. Gärtner J., Heydenreich M.: Annealed asymptotics for the parabolic Anderson model with a moving catalyst. *Stoch. Proc. Appl.* **116**, 1511–1529 (2006)
4. Gärtner J., den Hollander F.: Intermittency in a catalytic random medium. *Ann. Probab.* **34**, 2219–2287 (2006)
5. Gärtner J., den Hollander F., Maillard G.: Intermittency on catalysts: symmetric exclusion. *Electronic J. Probab.* **12**, 516–573 (2007)
6. Gärtner J., den Hollander F., Maillard G.: Intermittency on catalysts. In: Blath J., Mörters P., Scheutzw M. (eds.) *Trends in Stochastic Analysis*, London Mathematical Society Lecture Note Series 353, pp. 235–248. Cambridge University Press, Cambridge (2009)
7. Gärtner J., den Hollander F., Maillard G.: Intermittency on catalysts: three-dimensional simple symmetric exclusion. *Electronic J. Probab.* **72**, 2091–2129 (2009)
8. Gärtner J., den Hollander F., Maillard G.: Intermittency on catalysts: voter model. *Ann. Probab.* **38**, 2066–2102 (2010)
9. Gärtner J., König W.: The parabolic Anderson model. In: Deuschel J.-D., Greven A. (eds.) *Interacting Stochastic Systems*, pp. 153–179, Springer, Berlin (2005)
10. Gärtner J., Molchonov S.A.: Parabolic problems for the Anderson model. *Commun. Math. Phys.* **132**, 613–655 (1990)
11. Greven A., den Hollander F.: Phase transition for the long-time behavior of interacting diffusions. *Ann. Probab.* **35**, 1250–1306 (2007)
12. den Hollander, F.: *Large Deviations*. Fields Institute Monographs 14, American Mathematical Society, Providence, RI (2000)
13. Kesten H., Sidoravicius V.: Branching random walk with catalysts. *Electr. J. Probab.* **8**, 1–51 (2003)
14. Schnitzler, A., Wolff, T.: Precise asymptotics for the parabolic Anderson model with a moving catalyst or trap. In this volume.